Linear Algebra

1. A, B are square matrices with A + B = AB.

   Show that AB = BA .

   Proof :

   Let n = order of A.

   $\quad A + B = AB \cdots\cdots$ (i)

   $\Rightarrow AB - A - B = O_n$

   $\Rightarrow AB - A - B + I_n = I_n$

   $\Rightarrow (A - I_n)(B - I_n) = I_n$

   $\Rightarrow (B - I_n)(A - I_n) = I_n$ (why?)

   $\Rightarrow BA - A - B + I_n = I_n$

   $\Rightarrow BA = A + B \cdots\cdots$ (ii)

   (i) , (ii) $\Rightarrow$ AB = BA $\square$

2. M_1, M_2 are $n \times n$ matrices with $M_1M_2 = O, M_2 \neq O_n$.

   Show that det $M_1 = 0$.

   Proof :

   Suppose det $M_1 \neq 0$.

   Then M_1 is invertible.

   Let M_1^{-1} be the inverse of M_1

   $\quad M_1M_2 = O$

   $\Rightarrow M_1^{-1}M_1M_2 = O$

   $\Rightarrow IM_2 = O$

   $\Rightarrow M_2 = O \quad \rightarrow\leftarrow$ the assumption $M_2 \neq O_n$ $\square$

3. A, B are square matrices with $A^3 = B^3$, $A^2B = B^2A$, $A \neq B$.

   Show that $\det(A^2 + B^2) = 0$.

   Solution :

   $(A^2 + B^2)(A - B) = A^3 + B^2A - A^2B - B^3$

   $\qquad\qquad\qquad\qquad = O$ (by assumption)

   $\Rightarrow \det(A^2 + B^2) = 0$ ($\because A - B \neq O$, then apply Problem 2) $\square$

4. X, Y, Z are $n \times n$ matrices with $X + Y + Z = XY + YZ + ZX$.

   Show that the following are equivalent.

(1) $XYZ = XZ - ZX$

(2) $YZX = YX - XY$

(3) $ZXY = ZY - YZ$

Proof :

(I) $X + Y + Z = XY + YZ + ZX \cdots\cdots$ (i)

We show that (1) $\Rightarrow$ (2)

$$\begin{aligned}(X - I_n)(Y - I_n)(Z - I_n) &= XYZ - XY - XZ - YZ + X + Y + Z - I_n \\ &= (XYZ - XZ + ZX) \\ &\quad +(X + Y + Z - XY - YZ - ZX) - I_n \\ &= -I_n \qquad (\because (1)\ , (i)\ )\end{aligned}$$

$\Rightarrow (Y - I_n)(Z - I_n)(X - I_n) = -I_n$ (why?)

$\Rightarrow YZX - YZ - YX - ZX + Y + Z + X - I_n = -I_n$

$\Rightarrow YZX - YZ - YX - ZX + X + Y + Z = O_n$

$\Rightarrow YZX - YZ - YX - ZX + (XY + YZ + ZX) = O_n \qquad (\because (i)\ )$

$\Rightarrow YZX - YX + XY = O$

$\Rightarrow YZX = YX - XY$ $\square$

(II) In (I), we obtain

$$\left.\begin{aligned} X + Y + Z &= XY + YZ + ZX \\ XYZ &= XZ - ZX \end{aligned}\right\} \Rightarrow YZX = YX - XY.$$

Consider $X \to Y$, $Y \to Z$, $Z \to X$ in the above.

We have $$\left.\begin{aligned} Y + Z + X &= YZ + ZX + XY \\ YZX &= YX - XY \end{aligned}\right\} \Rightarrow ZXY = ZY - YZ.$$

Thus (2) $\Rightarrow$ (3)

Similarly (3) $\Rightarrow$ (1) $\square$

5. A is an $n \times n$ matrix with real entries.

Show that $\det(A^2 + I_n) \geq 0$.

$Proof_1$:

A is a real matrix

$\Rightarrow$ char. poly. of A is with real coefficients

$\Rightarrow$ the complex eigens of A come in pairs of conjugate numbers

Case 1: all eigen values of A are real.

Let $\lambda_1, \cdots, \lambda_n$ are eigen values of A, $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$

Then $\lambda_1^2 + 1, \cdots, \lambda_n^2 + 1$ are eigen values of $A^2 + I_n$

$$\Rightarrow \det(A^2+I_n) = (\lambda_1^2+1)\cdots(\lambda_n^2+1)$$
$$\geq 0 \qquad (\because \lambda_1,\cdots,\lambda_n \in \mathbb{R})$$

Case 2: all eigen values of A are in $\mathbb{C}-\mathbb{R}$.

Let $c_1, c_2, \cdots, \overline{c_1}, \overline{c_2}, \cdots$ are all eigen values of A, where $c_1, c_2 \in \mathbb{C}-\mathbb{R}$

$\Rightarrow c_1^2+1, c_2^2+1, \cdots, \overline{c_1}+1, \overline{c_2}+1, \cdots$ are eigen values of A^2+I_n

$$\Rightarrow \det(A^2+I_n) = (c_1^2+1)(c_2^2+1)\cdots(\overline{c_1}^2+1)(\overline{c_2}^2+1)\cdots$$
$$= (c_1^2+1)(c_2^2+1)\cdots(\overline{c_1^2+1})(\overline{c_2^2+1})\cdots$$
$$\geq 0 \qquad \text{(why?)}$$

Case 3: some eigen values of A are real and some are in $\mathbb{C}-\mathbb{R}$.

Let $\lambda_1, \lambda_2, \cdots, c_1, c_2, \cdots, \overline{c_1}, \overline{c_2}, \cdots$ are all eigen values of A,

where $\lambda_1, \cdots \in \mathcal{R}$, $c_1, \cdots \in \mathbb{C}-\mathbb{R}$

Then $\lambda_1^2+1, \lambda_2^2+1, \cdots, c_1^2+1, \cdots,$

$\overline{c_1}^2+1, \cdots$ are all eigen values of A^2+I_n

$$\Rightarrow \det(A^2+I_n) = (\lambda_1^2+1)\cdots(c_1^2+1)\cdots(\overline{c_1}^2+1)\cdots$$
$$= (\lambda_1^2+1)\cdots(c_1^2+1)\cdots(\overline{c_1^2+1})\cdots$$
$$\geq 0$$

$Proof_2$:

$$\det(A^2+I) = \det\left((A+iI)(A-iI)\right)$$
$$= \det\left((A+iI)\overline{(A+iI)}\right)$$
$$= \det(A+iI)\cdot \det\overline{(A+iI)}$$
$$= \det(A+iI)\cdot \overline{det(A+iI)} \geq 0 \qquad \square$$

6. Show that every permutation of $1, 2, \cdots, n$ $(n \geq 2)$ is either a cycle or a product of two cycles.

Proof :

Let π be a permutation of $1, 2, \cdots, n$.

Case 1: π is the identity function.

Then $\pi = (1\ 2)(1\ 2)$.

Case 2: π is not the identity function.

Then π can be written as a product of disjoint cycles.

Subcase 2.a: π is a cycle.

Nothing to prove.

Subcase 2.b: π is a product of at least two disjoint cycles.

Let $\pi = (a_{11}a_{12}\cdots a_{1r_1})(a_{21}a_{22}\cdots a_{2r_2})\cdots(a_{k1}a_{k2}\cdots a_{kr_k})$

be a product of disjoint cycles $(k \geq 2)$

$$\text{Then } \pi = (a_{k1}a_{k-1,1}\cdots a_{31}a_{21}a_{11})(a_{11}a_{12}\cdots a_{1r_1}a_{21}a_{22}\cdots a_{2r_2} \\ a_{31}a_{32}\cdots a_{3r_3}\cdots a_{k1}a_{k2}\cdots a_{kr_k}) \qquad \square$$

7. For $i = 1, 2, \cdots, n+1$, $A_i \in M_{1\times n}$ and $A_1 + A_2 + \cdots + A_{n+1} = O$.

For $i = 1, 2, \cdots, n$, let $M_i = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_{n+1} \end{bmatrix} \in M_{n\times n}$

Show that $\det M_1 = (-1)^{i-1} \det M_i$.

Proof :

$$\det M_1 = \det \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_{i-1} \\ A_i \\ A_{i+1} \\ \vdots \\ A_{n+1} \end{bmatrix} \qquad (\text{ by def. of } M_1)$$

$$= \det \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_{i-1} \\ -(A_1 + A_2 + A_3 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_{n+1}) \\ A_{i+1} \\ \vdots \\ A_{n+1} \end{bmatrix}$$

(by assumption, $A_1 + A_2 + \cdots + A_{n+1} = O$)

$$= \det \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_{i-1} \\ -A_1 \\ A_{i+1} \\ \vdots \\ A_{n+1} \end{bmatrix} \qquad (\text{ why? })$$

$$= (-1)^{i-2} \det \begin{bmatrix} -A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_{n+1} \end{bmatrix} \qquad \text{(why?)}$$

$$= (-1)^{i-1} \det M_i \qquad \square$$

8. Let M be an $n \times n$ matrix with every row sum and every column sum equal to 0.
   Let $M_{i,j}$ be the matrix obtained form M by deleting its i-th row and its j-th column.
   Show that for $1 \leq i , j , i' , j' \leq n$,

   $$(-1)^{i+j} \det M_{i,j} = (-1)^{i'+j'} \det M_{i',j'}.$$

   Proof :

   Since every column sum of M is 0, we have, by problem 7,

   $\det M_{1,j} = (-1)^{i-1} \det M_{i,j} \quad \cdots (1)$

   Similarly, since every row sum of M is 0, we have

   $\det M_{1,1} = (-1)^{j-1} \det M_{1,j} \quad \cdots (2)$

   $$\left.\begin{array}{l} \text{Thus } (1), (2) \Rightarrow (-1)^{i+j} \det M_{i,j} = \det M_{1,1} \\ \text{Similarly } (-1)^{i'+j'} \det M_{i',j'} = \det M_{1,1} \end{array}\right\} \Rightarrow$$

   $$(-1)^{i+j} \det M_{i,j} = (-1)^{i'+j'} \det M_{i',j'} \qquad \square$$

9. Suppose that $A_1, A_2, \cdots , A_k,\ B_1, B_2, \cdots , B_k$ are subsets of $\{1, 2, \cdots , n\}$.
   Let $L_1, L_2, \cdots , L_k,\ M_1, M_2, \cdots , M_k$ be the following polynomials
   in variables $x_1, x_2, \cdots , x_n$:

   $$L_i = \sum_{j \in A_i} x_j \quad , \quad M_i = \sum_{j \in B_i} x_j \qquad i = 1, 2, \cdots , k.$$

   Assume that $\displaystyle\sum_{1 \leq i < j \leq n} x_i x_j = \sum_{1 \leq i \leq k} L_i M_i$

   Show that $k \geq n - 1$.

   Proof :

   Consider the following linear equation (with real coefficients)
   in variables $x_1, x_2, \cdots , x_n$.

$$(*)\begin{cases} L_1 = 0 \\ L_2 = 0 \\ \quad\vdots \\ L_k = 0 \\ x_1 + x_2 + \cdots + x_n = 0 \end{cases}$$

Then the real solution of $(*)$ is the trivial solution.

($\because$ Suppose $x_1 = a_1,\ x_2 = a_2,\ \cdots,\ x_n = a_n$ is a real solution of $(*)$

$$\text{Then}\quad \begin{cases} L_1 = 0\ ,\ L_2 = 0\ ,\ \cdots\ ,\ L_k = 0 & \cdots(1) \\ a_1 + a_2 + \cdots + a_n = 0 & \cdots(2) \end{cases}$$

$$(1) \Rightarrow \sum_{1\le i<j\le n} a_i a_j = 0 \qquad \left(\because \sum_{1\le i<j\le n} x_i x_j = \sum_{1\le i\le k} L_i M_i\right)$$

$$\Rightarrow \quad 0 = (a_1 + a_2 + \cdots + a_n)^2 \qquad (\because (2))$$

$$= \sum_{1\le i\le n} a_i^2 + 2 \sum_{1\le i<j\le n} a_i a_j$$

$$= \sum_{1\le i\le n} a_i^2$$

$$\Rightarrow \quad a_1 = a_2 = \cdots = a_n = 0 \qquad (\because \text{each } a_i \text{ is real})$$)

Then $\quad k + 1 \ge n \qquad$ (why?)

$\Rightarrow \quad k \ge n - 1$ $\square$

10. Let A, B, C, D are metrices of orders $m \times m$, $n \times n$, $m \times n$, $m \times n$ respectively such that $AD - DB = C$.

    Show that $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$ and $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ are similar.

    Proof :

    $$AD - DB = C$$

    $$\Rightarrow \begin{pmatrix} I_m & D \\ O & I_n \end{pmatrix} \begin{pmatrix} A & C \\ O & B \end{pmatrix} \begin{pmatrix} I_m & -D \\ O & I_n \end{pmatrix} = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \qquad \text{(check!)}$$

    $\Rightarrow \begin{pmatrix} A & C \\ O & B \end{pmatrix}$ and $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ are similar.

    ($\because \begin{pmatrix} I_m & -D \\ O & I_n \end{pmatrix}$ is the inverse of $\begin{pmatrix} I_m & D \\ O & I_n \end{pmatrix}$) $\square$

11. A, D are nonsingular matrices and $M = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$. Find M^{-1}.

    Solution :

    Suppose that A is an $m \times m$ matrix, D is an $n \times n$ matrix.

    Let $\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_n \end{bmatrix}$, where E is an $m \times m$ matrix and

H is an $n \times n$ matrix.

$$\text{Then} \begin{cases} AE = I_m \\ AF + BH = O \\ DG = O \\ DH = I_n \end{cases}$$

$$\Rightarrow \left(\begin{array}{l} E = A^{-1},\ G = O\,,\ H = D^{-1}, \\ AF = -BH \end{array} \right.$$

$$\Rightarrow F = -A^{-1}BD^{-1}$$

$$\therefore\ M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}$$ □

12. A, B, C, D are $m_1 \times n_1$, $m_1 \times n_2$, $m_2 \times n_1$, $m_2 \times n_2$ matrices, and $m_1 + m_2 = n_1 + n_2$.

Show that $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{m_1m_2+n_1n_2} \det\begin{pmatrix} D & C \\ B & A \end{pmatrix}$.

Proof :

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{m_1m_2}\begin{pmatrix} C & D \\ A & B \end{pmatrix} \quad \text{(why?)}$$

$$= (-1)^{m_1m_2}(-1)^{n_1n_2}\begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

$$= (-1)^{m_1m_2+n_1n_2}\begin{pmatrix} D & C \\ B & A \end{pmatrix}$$ □

13. A, B, C, D are $n \times n$ matrices and C^t, D^t denote the transposes of C , D. Suppose that $CD^t + DC^t = O$.

Show that $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \det(AD^t + BC^t)^2$.

Proof :

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \det\begin{pmatrix} D & C \\ B & A \end{pmatrix} \quad \text{(by previous problem)}$$

$$= \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \det\begin{pmatrix} D & C \\ B & A \end{pmatrix}^t$$

$$= \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \det\begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}$$

$$= \det\begin{pmatrix} AD^t + BC^t & AB^t + BA^t \\ CD^t + DC^t & CB^t + DA^t \end{pmatrix}$$

$$
\begin{aligned}
&= \det\begin{pmatrix} AD^t + BC^t & AB^t + BA^t \\ O & CB^t + DA^t \end{pmatrix} \\
&= \det(AD^t + BC^t)\ \det(CB^t + DA^t) \\
&= \det(AD^t + BC^t)\ \det(CB^t + DA^t)^t \\
&= \det(AD^t + BC^t)\ \det(BC^t + AD^t) \\
&= \det(AD^t + BC^t)^2
\end{aligned}
$$

□

14. Show that

$$\sum_{k=1}^{m} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq k}}^{m} (1 - \frac{x_j}{x_k})} = 1 \quad \text{where } x_1, x_2, \cdots, x_m \ (m \geq 2) \text{ are distinct numbers.}$$

Proof$_1$:

The following is the well-known Vandermonde determinant.

$$\begin{vmatrix} x_1^{m-1} & x_2^{m-1} & \cdots & x_m^{m-1} \\ \vdots & \vdots & & \vdots \\ x_1^2 & x_2^2 & \cdots & x_m^2 \\ x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{1\leq i<j\leq m} (x_i - x_j) \tag{1}$$

Then

$$\begin{vmatrix} x_1^{m-1} & x_2^{m-1} & \cdots & x_m^{m-1} \\ \vdots & \vdots & & \vdots \\ x_1^2 & x_2^2 & \cdots & x_m^2 \\ x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$= x_1^{m-1} \begin{vmatrix} x_2^{m-2} & x_3^{m-1} & \cdots & \cdots & x_m^{m-2} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ x_2 & x_3 & \cdots & \cdots & x_m \\ 1 & 1 & \cdots & \cdots & 1 \end{vmatrix} - x_2^{m-1} \begin{vmatrix} x_1^{m-2} & x_3^{m-2} & x_4^{m-2} & \cdots & x_m^{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_3 & x_4 & \cdots & x_m \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \ x_1^{m-1} \prod_{\substack{1\leq i<j\leq m \\ i,j\neq 1}} (x_i - x_j) - x_2^{m-1} \prod_{\substack{1\leq i<j\leq m \\ i,j\neq 2}} (x_i - x_j) + \cdots\cdots\cdots\cdots\cdots \qquad \text{(by (1))}$$

$= \sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \prod_{\substack{1 \le i<j \le m \\ i,j \ne k}} (x_i - x_j).$

Combining the above with (1), we have

$$\sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \prod_{\substack{1 \le i<j \le m \\ i,j \ne k}} (x_i - x_j) \;=\; \prod_{1 \le i<j \le m} (x_i - x_j).$$

Thus we have

$$\sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \frac{\prod\limits_{\substack{1 \le i<j \le m \\ i,j \ne k}} (x_i - x_j)}{\prod\limits_{1 \le i<j \le m} (x_i - x_j)} = 1.$$

Equivalently,

$$\sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \frac{1}{\prod\limits_{k<j \le m} (x_k - x_j) \prod\limits_{1 \le i<k} (x_i - x_k)} = 1.$$

Thus

$$\sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \frac{1}{(-1)^{k-1} \prod\limits_{\substack{1 \le j \le m \\ j \ne k}} (x_k - x_j)} = 1,$$

which implies

$$\sum_{k=1}^{m} \frac{1}{\prod\limits_{\substack{1 \le j \le m \\ j \ne k}} (1 - \frac{x_j}{x_k})} = 1.$$

$Proof_2$:

Let $f(x_1, x_2, ..., x_m) = \sum_{k=1}^{m} (-1)^{k+1} \, x_k^{m-1} \prod_{\substack{1 \le i<j \le m \\ i,j \ne k}} (x_i - x_j)$. We need to show that $f(x_1, x_2, ..., x_m) = \prod_{1 \le i<j \le m} (x_i - x_j)$.

For $m = 2$, this is trivial.

For $m = 3$, this holds since

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2) \\ &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3). \end{aligned}$$

So let $m \ge 4$. We first show that $\prod_{1 \le i<j \le m} (x_i - x_j) | f(x_1, x_2, ..., x_m)$.

Suppose that p, q are positive integers with $1 \le p < q \le m$.

Then $f(x_1, x_2, ..., x_m) = (-1)^{p+1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq p}} (x_i - x_j) + (-1)^{q+1} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq q}} (x_i - x_j)$

$$+ \sum_{\substack{1\le k\le m \\ k\neq p,q}} (-1)^{k+1}\, x_k^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq k}} (x_i - x_j). \tag{1}$$

Note that

$$(-1)^{p+1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq p}} (x_i - x_j)$$

$$= (-1)^{p+1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i<q \\ i\neq p}} (x_i - x_q) \prod_{q<j\le m} (x_q - x_j)$$

$$= (-1)^{p+1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j)(-1)^{q-2} \prod_{\substack{1\le i<q \\ i\neq p}} (x_q - x_i) \prod_{q<j\le m} (x_q - x_j)$$

$$= (-1)^{p+q-1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i<q \\ i\neq p}} (x_q - x_i) \prod_{q<j\le m} (x_q - x_j)$$

$$= (-1)^{p+q-1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i\le m \\ i\neq p,q}} (x_q - x_i). \tag{2}$$

Note also that

$$(-1)^{q+1} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq q}} (x_i - x_j)$$

$$= (-1)^{q+1} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{1\le i<p} (x_i - x_p) \prod_{\substack{p<j\le m \\ j\neq q}} (x_p - x_j)$$

$$= (-1)^{q+1} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j)(-1)^{p-1} \prod_{1\le i<p} (x_p - x_i) \prod_{\substack{p<j\le m \\ j\neq q}} (x_p - x_j)$$

$$= (-1)^{p+q} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{1\le i<p} (x_p - x_i) \prod_{\substack{p<j\le m \\ j\neq q}} (x_p - x_j)$$

$$= (-1)^{p+q} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i\le m \\ i\neq p,q}} (x_p - x_i). \tag{3}$$

From (1),(2),(3), we see that

$$f(x_1, x_2, ..., x_m)$$

$$= (-1)^{p+q-1} x_p^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i\le m \\ i\neq p,q}} (x_q - x_i)$$

$$+ (-1)^{p+q} x_q^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\notin\{p,q\}}} (x_i - x_j) \prod_{\substack{1\le i\le m \\ i\neq p,q}} (x_p - x_i)$$

$$+ \sum_{\substack{1\le k\le m \\ k\neq p,q}} (-1)^{k+1}\, x_k^{m-1} \prod_{\substack{1\le i<j\le m \\ i,j\neq k}} (x_i - x_j).$$

Replacing x_p by x_q in $f(....., x_p, ..., x_q, ...)$, we obtain $f(....., x_q, ..., x_q, ...) = 0$, which implies $x_p - x_q | f(x_1, x_2, ..., x_m)$.

Hence $\prod_{1\le i<j\le m}(x_i-x_j)|f(x_1,x_2,......,x_m)$. (4)

Now we show that $f(x_1,x_2,......,x_m)=\prod_{1\le i<j\le m}(x_i-x_j)$.

Viewing $\prod_{1\le i<j\le m}(x_i-x_j)$ as a polynomial in the variable x_1, we see that the highest degree of x_1 is $m-1$, and the coefficient of x_1^{m-1} is $\prod_{2\le i<j\le m}(x_i-x_j)$; this means that

$$\prod_{1\le i<j\le m}(x_i-x_j)=(\prod_{2\le i<j\le m}(x_i-x_j))x_1^{m-1}+(\cdots\cdots\cdots)x_1^{m-2}+\cdots\cdots\cdots\cdots . \qquad (5)$$

Viewing $f(x_1,x_2,......,x_m)$ as a polynomial in the variable x_1, we see that the highest degree of x_1 is $m-1$, and the coefficient of x_1^{m-1} is $\prod_{\substack{1\le i<j\le m\\ i,j\neq 1}}(x_i-x_j)$; this means that

$$f(x_1,x_2,......,x_m)=(\prod_{\substack{1\le i<j\le m\\ i,j\neq 1}}(x_i-x_j))x_1^{m-1}+(\cdots\cdots\cdots)x_1^{m-2}+\cdots\cdots\cdots\cdots . \qquad (6)$$

From (4),(5),(5), we have that $f(x_1,x_2,......,x_m)=\prod_{1\le i<j\le m}(x_i-x_j)$. This completes the proof.